Residue currents for holomorphic foliations

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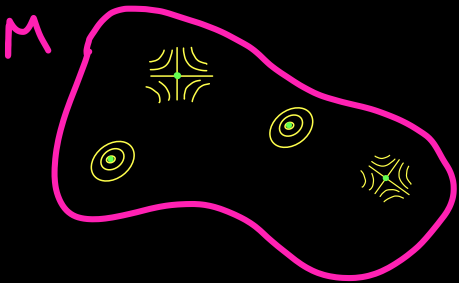
Theorem [Poincaré-Hopf]: M: compact smooth manifold, $\mathcal{V}$: vector field on M with isolated singularities ($=$ isolated zeroes). Then

$$\sum_{p \in \{\mathcal{V}=0\}} \mathrm{index}_p(\mathcal{V}) = \chi(M)$$

- $\mathrm{index}_p(\mathcal{V})$ is a number depending only on the local behavior of $\mathcal{V}$ around p



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Today: analogue for holomorphic foliations on a complex manifold

Baum-Bott Theorem, *1972*

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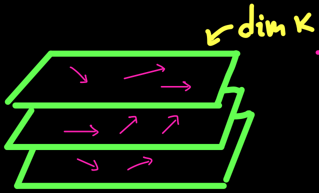
Frobenius Theorem:

LOCALLY :



$T\mathscr{F}$ = horizontal distribution

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Proof: existence of good connections on $N\mathscr{F}$, compatible with both the holomorphic and foliated structure

Def.: $\varphi_0 : TM \to N\mathscr{F} = TM/T\mathscr{F}$ canonical projection

D connection on $N\mathscr{F}$ is basic if of type $(1,0)$ and

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Remark: $\operatorname{codim} \text{sing}\mathscr{F} \geqslant 2$

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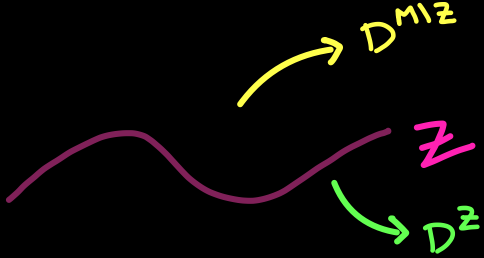

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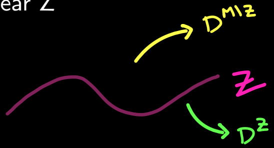

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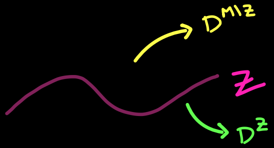

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Then

$$c(N\mathscr{F}) := c\left(\sum_{i=0}^{N} (-1)^i E_i\right) = \prod_{i=0}^{N} c(E_i)^{(-1)^i}$$

and similarly for Chern forms

Recall: $\mathrm{Res}^{\phi}(\mathscr{F};Z) \in H^{2\ell}(M,\mathbb{C})$ such that $\sum_{Z\subset \mathrm{sing}\,\mathscr{F}} \mathrm{Res}^{\phi}(\mathscr{F};Z) = \phi(N\mathscr{F})$

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$$\mathrm{Res}^{\phi}(\mathscr{F}; p) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{|a_1|=\varepsilon, \ldots, |a_n|=\varepsilon} \phi\left(\left(\frac{\partial a_i}{\partial z_j}\right)_{i,j}\right) \frac{dz_1 \wedge \ldots \wedge dz_n}{a_1 \cdots a_n}$$

(Grothendieck residue)

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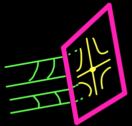

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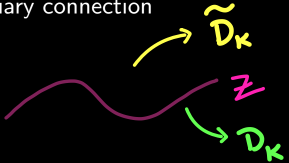
where $\chi \sim \mathbf{1}_{[1,+\infty]}$ and D_k auxiliary connection



⋆ By Chern-Weil $c_j(N\mathscr{F})$ represented by

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Proposition [KLW]: If connections have "good" singularities, then

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is a well defined current.

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$$D_0 := D^{TM} - (\mathcal{D}\varphi_1)\,\sigma_1(dz \cdot \partial/\partial z)$$

and

$$D_{basic}(\varphi_0 v) := -\varphi_0 D_0(\pi_0 v),$$

where $\pi_0 = I - \varphi_1\sigma_1$ orthogonal proj. from TM onto $(\text{Im}\varphi_1)^\perp = (T\mathscr{F})^\perp$

Once we have D_{basic} on $N\mathscr{F}$, can use φ_k and σ_k to induce compatible connections $\widetilde{D}_k$ on E_k.

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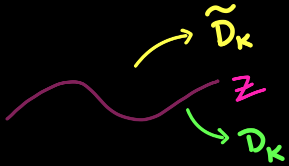

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Multivariable residue theory: Andersson, Coleff, Herrera, Lärkäng Passare, Samuelsson-Kalm, Tsikh, Wulcan, Yger, etc.

Thank you!